United Kingdom Mathematics Trust

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British Mathematical Olympiad Round 1 2019

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

Olympiad marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for is full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has a viable overall strategy or not. An answer which is essentially a solution will be awarded near maximum credit, with marks deducted for errors of calculation, flaws in logic, omission of cases or technical faults. On the other hand, an answer which does not present a complete argument is marked on a '0 plus' basis; a small number of marks (often capped at 3) might be awarded for particular cases or insights.

This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore important that candidates taking Olympiad papers realise the importance of the comment in the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

General comments

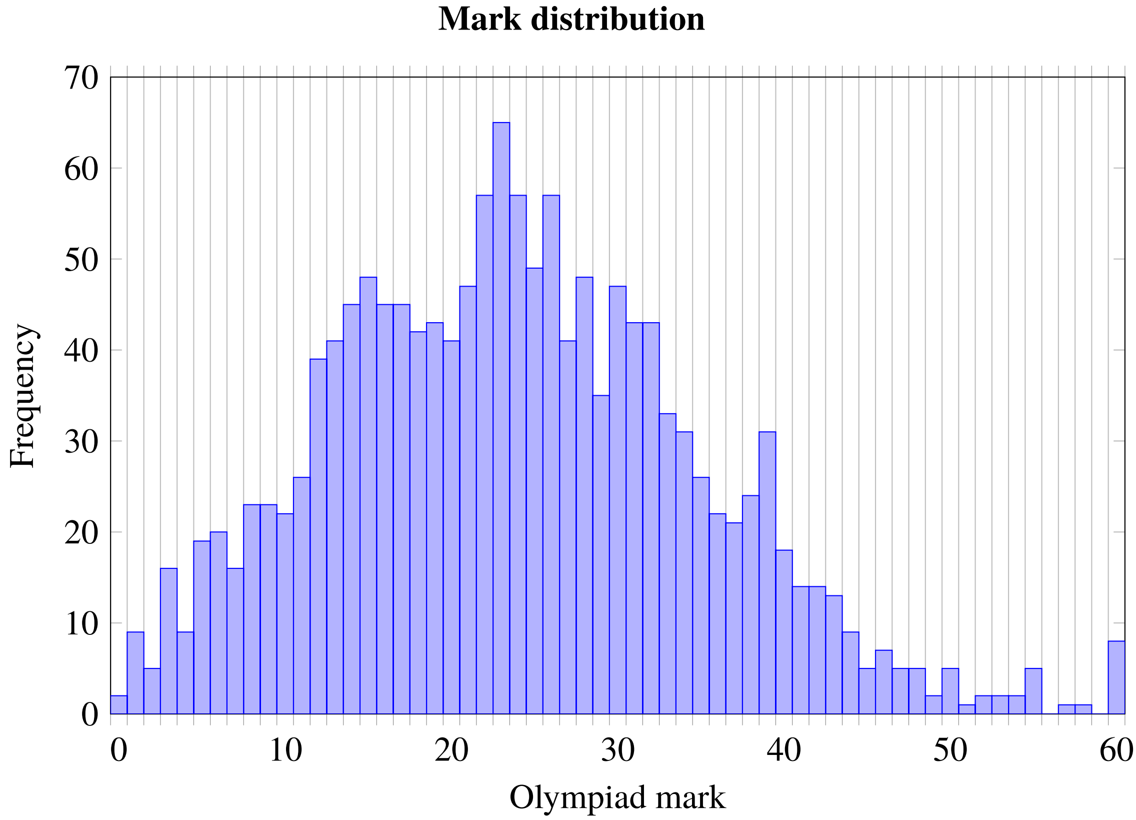
Candidates engaged very well with this year's paper, and the majority made substantial progress on at least two of the problems. It was encouraging to see a number scoring highly on question 3, which hopefully served as a reminder that BMO1 geometry problems often do not require much beyond GCSE circle theorems (and their converses) together with some ingenuity. In questions 4 and 5 many candidates found correct numerical answers, but failed to adequately explain their reasoning. In question 4 there was often a lack of precision when using the word 'behind' (does it mean *directly* behind or *somewhere* behind?), while in question 5 many candidates showed that multiples of 28 gave perfect arrangements without realising that they needed to show that no other numbers did. Question 6 was arguably as hard as it has ever been, and only the very best candidates came close to solving it.

A few scripts suffered from a lack of words explaining where various symbolic expressions came from, but these were reassuringly rare. Many more scripts gave copious discursive commentary, and would have benefited from breaking up their prose into separate claims and proofs, and by being a little more concise in places.

The 2019 BMO1 paper was sat by 1475 candidates. The scripts were marked in Cambridge on 7th and 8th December by a team of Oscar Arevalo, Eszter Backhausz, Tibor Backhausz, Connie Bambridge-Sutton, Sam Bealing, Emily Beatty, Jonathan Beckett, Cathy Beckett, Phillip Beckett, Jamie Bell, Lex Betts, Robin Bhattacharyya, Tom Bowler, Andrew Carlotti, Philip Coggins, Arthur Conmy, James Cranch, Alex Darby, Ronan Flatley, Richard Freeland, Tony Gardiner, James Gazet, Daniel Griller, Alex Gunning, Stuart Haring, Tim Hennock, Liam Hill, Ina Hughes, Ian Jackson, Shavindra Jayasekera, Vesna Kadelburg, Hadi Khan, Patricia King, Warren Li, Harry Metrebian, Joseph Myers, Eve Pound, Melissa Quail, Zarko Randelovic, Dominic Rowland, Amit Shah, Jerome Watson, Patrick Winter, Harvey Yau and Renzhi Zhou.

The problems were proposed by Nick MacKinnon, Tom Bowler, Daniel Griller, Daniel Griller, Dominic Rowland and Sam Bealing respectively.

In addition to the written solutions in this report, video solutions can be found here.



Candidates with scores ≥ 40 were invited to sit BMO2.

Question 1

Show that there are at least three prime numbers p less than 200 for which $p + 2$, $p + 6$, $p + 8$ and $p + 12$ are all prime. Show also that there is only one prime number q for which $q + 2$, $q + 6$, $q + 8$, $q + 12$ and $q + 14$ are all prime.

Solution

For the first part, $p = 5$, 11 and 101 all work, since all the numbers

$$\begin{gathered} 5, 7, 11, 13, 17 \\ 11, 13, 17, 19, 23 \\ 101, 103, 107, 109, 113 \end{gathered}$$

are all prime.

For the second part, $q = 5$ works, since the first sequence above can be continued on with 19. This is the only possibility. Clearly $q \neq 2$, and the odd numbers q, $q + 12$, $q + 14$, $q + 6$ and $q + 8$ cover all possible odd last digits, so one must end in a 5. Thus one of these numbers must be *equal* to 5, since they are all prime, and the only possibility is that $q = 5$. (If one uses instead the list q, $q + 2$, $q + 14$, $q + 6$, $q + 8$, one needs to rule out the possibility that $q + 2 = 5$ by, for example, observing that if $q + 2 = 5$, then $q + 6 = 9$ is not prime.)

Markers' comments

The vast majority of candidates attempted this question, and most made good progress. For the first part the three values of p given in the solution are the only valid ones less than 200 (the next is $p = 1481$) and a number of candidates provided additional incorrect values of p. In particular the following numbers were often mistaken for primes: $91 = 7 \times 13$, $119 = 7 \times 17$, $143 = 11 \times 13$, $161 = 7 \times 23$, $169 = 13 \times 13$ and $203 = 7 \times 19$.

In the second part many candidates made sensible use of last digits or arithmetic modulo 5. These candidates generally went on to solve the problem. A common minor error was to assert that some number ending in 5 cannot be prime, forgetting 5 itself. This happened most often with the prime $q + 2$, meaning that some candidates didn't successfully eliminate the single possibility that $q = 3$. The most common major error was to assume that the condition $p < 200$ from the first part also applied to q in the second. Candidates who did this and then listed all small possibilities for q were heavily penalised. However, those who gave arguments covering one, two and three digit numbers could still score nearly full marks if their arguments were easy to generalise.

Remark

It is conjectured that there are infinitely many primes which satisfy the requirements of the first part of this question. This claim is a strengthening of the famous 'Twin Primes' conjecture which has been intensively studied for centuries. There has been significant progress since 2013, but the proof or disproof of this conjecture remains beyond the reach of current mathematical techniques.

Question 2

A sequence of integers $a_1, a_2, a_3, \ldots$ satisfies the relation:

$$4a_{n+1}^2 - 4a_n a_{n+1} + a_n^2 - 1 = 0$$

for all positive integers n. What are the possible values of a_1?

SOLUTION

We start by factorising the expression involving the a_i to obtain $(2a_{n+1} - a_n)^2 - 1 = 0$.

From here we can either add 1 to both sides and take a square root to find $(2a_{n+1} - a_n) = \pm 1$, or factorise the difference of two squares to see that $(2a_{n+1} - a_n - 1)(2a_{n+1} - a_n + 1) = 0$.

Either way we have $a_{n+1} = \frac{1}{2}(a_n \pm 1)$. (This can also be obtained by viewing the equation as a quadratic in a_{n+1}, using the quadratic formula and simplifying.)

If a_1 is even then a_2 will not be an integer.

If a_n is odd, then the two possible values of a_{n+1} are consecutive integers. Thus we can choose a_{n+1} to be an odd integer. In particular, for any odd integer a_1 there exists a possible sequence consisting entirely of odd integers.

ALTERNATIVE

As before we have $a_{n+1} = \frac{1}{2}(a_n \pm 1)$.

If $a_n > 1$ then $a_n > a_{n+1} \geq 1$ so if a_1 is positive, then the terms decrease until they reach 1. If $a_n < -1$ then $a_n < a_{n+1} \leq -1$, so if a_1 is negative, then the terms increase until they reach -1.

As $a_n = 2a_{n+1} \pm 1$ we may construct the sequence in reverse from $a_k = \pm 1$ back to a_1. We obtain $a_1 = \pm 2^{k-1} \pm 2^{k-2} \pm \ldots \pm 2^1 \pm 1$ and so a_1 must be odd. We can prove by (complete) induction on the magnitude of a_1 that a_1 can be any odd number. Suppose we want to construct a sequence with $a_1 = 2k + 1 = 2(k + 1) - 1$. One of k and $k + 1$ is an odd number of smaller magnitude, so, by induction, there is a legal sequence with $a_1 = k$ or $a_1 = k + 1$. We may take this sequence, increase all the subscripts by 1 and insert the desired value of a_1 at the beginning.

MARKERS' COMMENTS

There were many excellent responses to this question. Both solution strategies outlined above were used successfully by candidates, though the second was much rarer.

This question also threw up several misconceptions that led to candidates scoring a maximum of 3 marks. The most common was to assume that, when faced with the two expressions for a_{n+1}, there were then only two ways to generate a sequence, either by repeatedly using $a_{n+1} = \frac{1}{2}(a_n + 1)$ or repeatedly using $a_{n+1} = \frac{1}{2}(a_n - 1)$. Candidates making this mistake generally went on to show that, if the choice of sign was not allowed to vary within the sequence, the only possible values of a_1 are ± 1.

The second most prevalent misconception that arose was for candidates to assume that $(2a_{n+1} - a_n)^2 = 1$ led to $a_{n+1} = \frac{1}{2}(a_n + 1)$ only. It is important for candidates to appreciate that $x^2 = 1$ may not mean $x = \sqrt{1}$ and it is interesting to note that for this problem such an error affected the solution set significantly.

Many solutions gave the correct set of values for a_1 but still lost a couple of marks, due to a lack of justification of two important observations. The first of was that a_1 could not be even, which some candidates stated but did not explain and other omitted to mention. The second related to the fact that the two possible values for a_{n+1}, given an odd a_n, had different parity. This is a crucial observation needed in order to continue the sequence and some stated it as fact with no proof.

Some candidates attempted to apply induction once they had expressions for a_{n+1}, but many were unclear about the statement they were attempting to prove. Others attempted to start with $a_{n+1} = \pm 1$ and work backwards, iterating both expressions for a_{n+1} to create a tree of possible values for a_1. These candidates often observed various patterns in their trees and claimed, without proof, that these patterns continued indefinitely. Such scripts were awarded a maximum of 3 marks.

Some candidates assumed the terms of the sequence were positive, this was not stated in the question and led to a deduction of 1 mark.

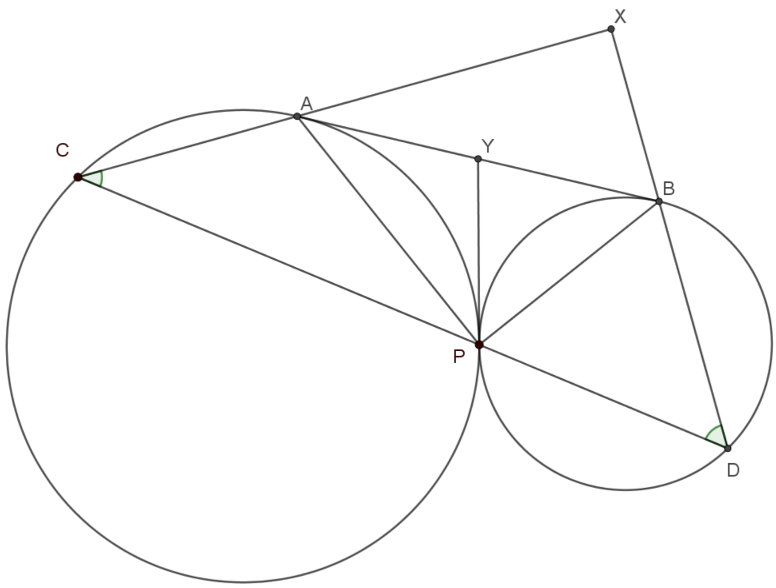
Question 3

Two circles S_1 and S_2 are tangent at P. A common tangent, not through P, touches S_1 at A and S_2 at B. Points C and D, on S_1 and S_2 respectively, are outside the triangle APB and are such that P is on the line CD.

Prove that AC is perpendicular to BD.

Solution

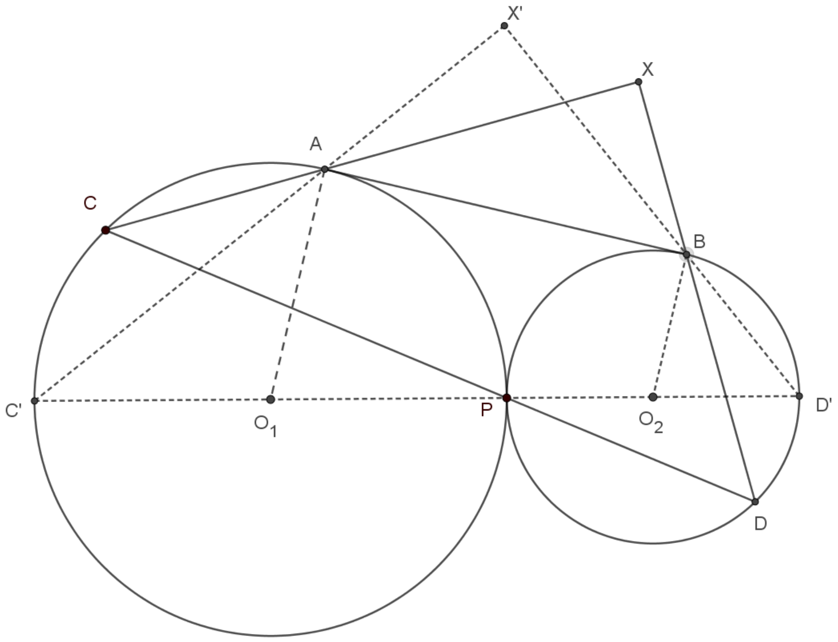
Let AC and BD intersect at X, and let the common tangent at P intersect AB at Y as shown.



Let $\angle DCA = \gamma$ and $\angle BDC = \delta$. By the alternate segment theorem $\angle YPA = \angle PAY = \gamma$ and $\angle YBP = \angle BPY = \delta$. The angles in triangle BPA sum to $2\gamma + 2\delta$ so $\gamma + \delta = 90°$.

Now looking at the angles in triangle DXC shows that $\angle DXC = 90°$ as required.

Alternative



We consider the special case where $C'P$ and PD' are diameters of the two circles. It is clear that C', O_1, P, O_2 and D' are collinear, since the radii O_1P and O_2P are both perpendicular to the common tangent at P.

The tangent AB is perpendicular to the radii AO_1 and BO_2, so considering the angles in the quadrilateral AO_1O_2B we see that $\angle O_2O_1A + \angle BO_2O_1 = 180°$. However these are external angles of the isosceles triangles AO_1C' and BO_2D' so we see see that $\angle O_1C'A + \angle BD'O_2 = 90°$. Considering triangle $D'C'X'$ shows that $\angle C'X'D' = 90°$.

To prove the result for an arbitrary line CD through P we observe that $\angle BDP = \angle BD'P$ and $\angle PCA = \angle PC'A$ by angles in the same segment, and the result follows.

Markers' comments

We were pleased to see many excellent solutions to this problem. Most candidates proceeded along the lines of the first solution. A common variation was to use the quadrilateral O_1O_2AB from the second solution and isosceles triangles AO_1P and BO_2P to show that $\angle APB = 90°$ and then proceed as in the first solution. Yet another possibility was to start by showing that $\angle PO_1A + \angle BO_2P = 180°$ and then use the fact that the angle at the centre of a circle is twice the angle at the circumference and therefore $\angle PCA + \angle BDP = 90°$.

Many of the solutions were very well explained. However, some candidates still produced long lists of angle calculations with no explanations, and they were penalised for this. It is important to make it clear at each step which circle theorem, or which triangle is being used, although the standard GCSE theorems may be used without proof.

A mark was often lost for using the fact that O_1PO_2 is a straight line without explicitly stating it; it should be noted that this is only the case because P is the point of tangency of the two circles. A more serious omission was to claim, without proof, that $\angle APB = 90°$; this was heavily penalised.

In geometry problems, candidates often only consider special cases. In this question, a number of candidates only considered the case when the line CD passes through the centres of the two circles, which could score the marks available for the first part of the alternative solution. Some candidates assumed further that the two circles are equal and argued by symmetry; this approach usually earned no marks.

Remark

The condition that points C and D lie outside triangle APB restricts us to essentially one diagram, but the result is true more generally. Checking the other cases is fairly straightforward, but would be a worthwhile exercise.

Question 4

There are 2019 penguins waddling towards their favourite restaurant. As the penguins arrive, they are handed tickets numbered in ascending order from 1 to 2019, and told to join the queue. The first penguin starts the queue. For each $n > 1$ the penguin holding ticket number n finds the greatest $m < n$ which divides n and enters the queue directly behind the penguin holding ticket number m. This continues until all 2019 penguins are in the queue.

(a) How many penguins are in front of the penguin with ticket number 2?

(b) What numbers are on the tickets held by the penguins just in front of and just behind the penguin holding ticket 33?

Solution

We begin by noting that the largest $m < n$ which divides n is equal to n divided by its smallest factor (other than 1). Clearly this smallest factor must be prime.

For part (a) we claim that the penguins that end up somewhere behind 2 are precisely the larger powers of 2. Penguin 3 goes in front of 2, so the claim holds when only three penguins have arrived. Now suppose the claim holds when $k - 1$ penguins have arrived and consider penguin k. If k is a power of 2, then its largest proper factor is also a power of 2, so it goes directly behind that factor and hence somewhere behind 2. If k is prime it goes directly behind 1 and thus somewhere in front of 2. Finally, if k has smallest prime factor p and another prime factor $q > 2$, then k/p is a multiple of q and not a power of 2. Penguin k goes directly behind k/p and we already know k/p is somewhere in front of 2, so k goes somewhere in front of 2. The claim implies that when 2019 penguins have arrived, only penguins 4, 8, 16, 32, 64, 128, 256, 512 and 1024 are behind 2. So the remaining 2009 penguins are in front of penguin 2.

For part (b) we first consider the penguins who stand directly in front of penguin 33 at some stage in the queuing process. When 33 arrives 11 is directly in front of it and 22 is behind it. The next multiple of 11, namely 44, stands behind 22. Later 55 comes and stands directly in front of 33. The next penguin to come between 55 and 33 is the next available multiple of 55, namely 110. Now we forget about penguin 55 and focus on who comes in between 110 and 33. It is the next available multiple 110, namely 220. Continuing in this way we see that 440, then 880 and finally 1760 occupy the spot directly in front of 33.

Finally we turn to the penguins who stand directly behind 33 at some stage. Their numbers must be of the form $33k$ for some k. However, if $k > 3$ then $11k > 33$ so the only numbers that ever stand directly behind 33 and $33k$ for $k \leq 3$. On arrival, penguin 66 stands directly behind 33, but is later replaced by penguin 99, who stays directly behind 33 from then on.

Markers' comments

We were pleased with candidates' willingness to have a go at this problem, and all parts of the question were found quite accessible.

In part (a), many candidates observed that powers of 2 end up behind penguin 2 by working out some small cases. However, to score well they needed an argument for why this pattern continues. In particular, they needed to show both that powers of 2 end up behind 2 and that no other penguin does. Some idea of induction is very helpful for stating this formally, but plenty of candidates also got the marks for a detailed verbal description without using induction explicitly.

In part (b) a common mistake was thinking that penguin 44 entered in front of 33 and this error was heavily penalised. Candidates were surprisingly careless in finding the penguin behind 33 and many thought that the answer was 33^2 or 33×31. We would advise checking this kind of numerical answer carefully using some smaller versions of the same problem. Even candidates finding the correct answer of 99 could score few marks for this part if they did not explain why no higher penguin entered behind 33; there were a number of excellent solutions to this part.

Question 5

Six children are evenly spaced around a circular table. Initially, one has a pile of $n > 0$ sweets in front of them, and the others have nothing. If a child has at least four sweets in front of them, they may perform the following move: eat one sweet and give one sweet to each of their immediate neighbours and to the child directly opposite them. An arrangement is called *perfect* if there is a sequence of moves which results in each child having the same number of sweets in front of them. For which values of n is the initial arrangement perfect?

Solution

The initial arrangement is perfect if and only if n is divisible by 28. We number the children from 1 to 6 around the table, and assume child 1 starts with the sweets.

Suppose $n = 28k$, for $k > 0$, and consider the following sequence of moves:

(1) Child 1 makes $7k$ moves. This leaves all the odd-numbered children without any sweets, and all of the even-numbered children with $7k$ sweets.

(2) Each even-numbered child now make k moves. This leaves each child with $3k$ sweets so the arrangement is perfect.

It remains to show that if the initial arrangement is perfect, then n must be divisible by 28. There are various possible approaches.

Call the odd-numbered children 'Team O' and the even-numbered children 'Team E'. Now consider the difference between the total number of sweets held by Team O and the total number held by Team E. Initially this difference is n. Once all children have the same number of sweets this difference is 0, and any move changes the difference by exactly 7. Thus 7 divides n.

Next consider the difference between the number of sweets held by child 1 and the number held by child 3. At the start this difference is n and at the end this difference is zero. Moves by children 2, 4, 5 and 6 do not change this difference, while moves by children 1 and 3 change it by exactly four each time. Thus 4 divides n.

Since n is a multiple of 4 and 7, we conclude that it must be a multiple of 28 as required.

Alternative

Suppose child 1 makes a total of a moves, child 2 makes a total of b moves and so on. After all moves have been made, the number of sweets each child has are given in the table below. We are interested in the case when all these quantities are equal.

Child	Number of sweets
1	$n + b + d + f - 4a$
2	$a + c + e - 4b$
3	$b + d + f - 4c$
4	$a + c + e - 4d$
5	$b + d + f - 4e$
6	$a + c + e - 4f$

Equating the number of sweets that children 2, 4 and 6 have:

$$a + c + e - 4b = a + c + e - 4d = a + c + e - 4f,$$

so

$$b = d = f. \tag{1}$$

Similarly, equating the number of sweets that children 1, 3 and 5 have gives:

$$b + d + f + n - 4a = b + d + f - 4c = b + d + f - 4e$$

so

$$e = c \text{ and } n = 4(a - c). \tag{2}$$

Now, equating the totals for children 2 and 3 and plugging in (1) and (2) shows that:

$$a + 2c - 4b = 3b - 4c \text{ or } a = 7b - 6c. \tag{3}$$

Plugging (3) into (2) gives $n = 28(b - c)$. .

Alternative

It is tempting to make claims like 'the order of moves does not matter' or 'children 3 and 5 should never make any moves'. These claims are both false, but the ideas can be captured in a more careful argument.

Suppose there a sequence of moves $\mathcal{S}$ showing that the initial arrangement with n sweets is perfect, and that in that sequence child 1 makes a moves, child 2 makes b and so on. Since children 2, 4 and 6 start with the same number of sweets and only ever gain them simultaneously, they must all make the same number of moves, so $b = d = e$. Similarly, children 3 and 5 gain sweets simultaneously, so must make the same number moves giving $c = e$.

Now we claim that the following alternative sequence $\mathcal{T}$ of moves can also be used to show that the initial arrangement is perfect:

(1) Child 1 makes $a - c$ moves.

(2) Children 2, 4 and 6 each make $b - c$ moves.

(3) Children 3 and 5 make no moves.

In sequence $\mathcal{T}$ each child makes exactly c fewer moves than they do in $\mathcal{S}$, so each ends up with c more sweets. However, to actually prove our claim about $\mathcal{T}$, we must also show it is valid sequence of moves. In particular we must check that $a - c$ and $b - c$ are positive, and that no child ever has a negative number of sweets.

In sequence $\mathcal{S}$ each move by child 3 requires a total of at least four prior moves by children 2, 4 and 6. Thus $3b \geq 4c$ or $b - c \geq \frac{c}{3}$ which implies that $b - c > 0$ since even if $c = 0$ we must have $b > 0$. Similarly, each move by child 2 requires at least four prior moves by odd-numbered children, so $a + 2c \geq 4b$ which gives $a - c \geq 4b - 3c > 0$.

It remains to check that no child ever has a negative number of sweets in $\mathcal{T}$. Once child 1 has made all their moves, they gain sweets at the same time as child 3. Since they end up with the same number, child 1 must have exactly zero sweets when they finish their moves, and after that they only gain sweets. Children 2, 4 and 6 gain sweets steadily until they start making moves. After that they lose sweets until they reach the (positive) number finally held by each child. Children 3 and 5 never lose any sweets, so the claim is established.

We have already observed that $n - 4(a - c)$ must equal zero. Also, at the end of sequence $\mathcal{T}$ each child has $3(b - c)$ sweets and the total number of moves made was $(a - c) + 3(b - c)$. Thus $n = 6 \times 3(b - c) + (a - c) + 3(b - c)$ so $\frac{3}{4}n = 21(b - c)$ giving $n = 28(b - c)$.

Markers' comments

Lots of students did a good job showing that a configuration where one child starts with 28 sweets is perfect, and indeed that the same was true if the starting number was a multiple of 28. Unfortunately, a substantial number of students asserted that these were the only possibilities because their strategy that worked when n is a multiple of 28 would not work if n were not a multiple of 28. These students failed to consider why *no* sequence of moves would allow all children to end up with the same number of sweets from the initial configuration.

There were two approaches that students usually managed to see through:

- Setting up algebra as in the first alternative solution, and calmly working through it;
- Trying some examples and realising that the "odd" and "even" teams of the official solution are useful.

Unfortunately, a large number of students tried to argue that the order of moves is irrelevant. If this is assumed, the algebra can be simplified, but proving that this assumption does not eliminate any perfect initial configurations is quite tough, so these students missed the crux of the problem. Correct solutions along the lines of the second alternative were extremely rare.

Remark

The numbers 6 and 28 are both called *perfect* numbers since they are equal to the sum of their proper divisors. Euler showed that every even perfect number must be of the form $2^{m-1}(2^m - 1)$ where $2^m - 1$ is a (Mersenne) prime. However, it is not currently known whether there are infinitely many perfect numbers, or indeed whether any odd perfect numbers exist.

Question 6

A function f is called *good* if it assigns an integer value $f(m,n)$ to every ordered pair of integers (m,n) in such a way that for every pair of integers (m,n) we have:

$$2f(m,n) = f(m-n, n-m) + m + n = f(m+1, n) + f(m, n+1) - 1.$$

Find all good functions.

Solution

We write $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{R}$ for the left, middle and right parts of the displayed equations.

Note first that, by substituting $m = n = 0$ into $\mathcal{L} = \mathcal{M}$, we get $f(0,0) = 0$.

Now, writing $g(m)$ for $f(m, -m)$, the equation $\mathcal{L} = \mathcal{M}$ gives $f(m,n) = \frac{1}{2}(g(m-n) + m + n)$. Making this substitution on equation $\mathcal{L} = \mathcal{R}$ gives

$$g(m-n) + m + n = \frac{1}{2}(g(m+1-n) + m + 1 + n) + \frac{1}{2}(g(m-n-1) + m + n + 1) - 1,$$

and taking $n = 0$ then gives

$$g(m) + m = \frac{1}{2}(g(m+1) + m + 1) + \frac{1}{2}(g(m-1) + m + 1) - 1,$$

which simplifies to $g(m) = \frac{1}{2}(g(m+1) + g(m-1))$.

This says that $g(m)$ is linear; since we determined at the start that $g(0) = 0$, we get $g(m) = am$, and hence $f(m,n) = \frac{1}{2}((1+a)m + (1-a)n)$. By taking $m = 1, n = 0$, we see that a should be odd; taking $a = 2b + 1$ gives $f(m,n) = (b+1)m - bn$. It can readily be checked that this works for any value of b.

Alternative

Using $\mathcal{L} = \mathcal{M}$ for $f(m+1, n+1)$ and for $f(m,n)$ gives

$$2f(m+1, n+1) = f(m-n, n-m) + m + n + 2,$$
$$2f(m,n) = f(m-n, n-m) + m + n.$$

Hence $f(m+1, n+1) = f(m,n) + 1$, and so, by induction in both directions, we have $f(m,n) = f(m-n, 0) + n$.

Substituting this into the original equations, and subtracting $2n$ from all sides, gives us

$$2f(m-n, 0) = f(2m-2n, 0) = f(m-n+1, 0) + f(m-n-1, 0).$$

Writing p for $m - n$, we have

$$2f(p,0) = f(2p, 0) = f(p+1, 0) + f(p-1, 0).$$

From these we deduce (similarly to Solution 1) that $f(p,0) = bp$ for some b, and this gives $f(m,n) = b(m-n) + n$; it can readily be checked that all such solutions work.

Markers' comments

This question was not attempted by many students; among the attempts we saw, many weren't successful. We suspect that most successful students mixed the following three strategies, allowing their experiences with each to inform their attempts at the others:

(1) Using small cases of the recurrence relation to try to deduce things about $f(m,n)$ for m and n small integers. (Looking out for helpful things is easier if one has attempted strategies (2) and (3).)

(2) Attempting clever substitutions, to try to say helpful general things about the function. (Precisely what things are helpful can best be told by attempting strategies (1) and (3).)

(3) Attempting to think of solutions. (Indeed, trying solutions like $f(m,n) = am + bn + c$ may not come naturally at first, but knowing which solutions to try does come naturally if one has made attempts at strategies (1) and (2).)

In contrast, students who focused their attention on only one of these three strategies were likely to grind to a halt sooner or later.

